Lectures on BRS invariance for massive boson fields

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Abstract

These notes correspond to lectures given at the Villa de Leyva Summer School in Colombia (July 2007). Our main purpose in this short course on BRS invariance of gauge theories is to illuminate corners of the theory left in the shade by standard treatments. The plan is as follows. First a review of Utiyama's "general gauge theory". Promptly we find a counterexample to it in the shape of the massive spin-1 Stückelberg gauge field. This is not fancy, as the massive case is the most natural one to introduce BRS invariance in the context of free quantum fields. Mathematically speaking, the first part of the course uses Utiyama's notation, and thus has the flavour and non-intrinsic notation of standard physics textbooks. Next we deal with boson fields on Fock space and BRS invariance in connection with the existence of Krein operators; the attending rigour points are then addressed.

Contents

1 Utiyama's method in classical gauge theory

1.1 A historical note

Ryoyu Utiyama developed non-abelian gauge theory early in 1954 in Japan, almost at the same time that Yang and Mills [1] did at the Princeton's Institute for Advanced Study (IAS), that Utiyama was to visit later in the year. Unfortunately, Utiyama chose not to publish immediately, and upon his arrival at IAS on September of that year, he was greatly discouraged to find he had apparently just been "scooped".

In fact, he had not, or not entirely. He writes: "(In March 1955), I decided to return to the general gauge theory, and took a closer look at Yang's paper, which had been published in 1954. At this moment I realized for the first time that there was a significant difference between Yang's theory and mine. The difference was that Yang had merely found an example of non-abelian gauge theory whereas I had developed a general idea of gauge theory that would contain gravity as well of electromagnetic theory. Then I decided to publish my work by translating it into English, and adding an extra section where Yang's theory is discussed as an example of my general theory" [2].

Utiyama's article appeared on the March 1, 1956 issue of the Physical Review [3], and is also is reprinted in the book by the late Lochlainn O'Raifeartaigh [2], where the foregoing (and other) interesting historical remarks are made.

As Utiyama himself does above, most people who read his paper focused on the kinship there shown between gravity and gauge theory. This is in

some sense a pity, because in contrast with "textbook" treatments of Yang–Mills theories —see [4] for just one example— which manage to leave, despite disguises of relatively sophisticated language, a strong impression of arbitrariness, Utiyama strenously tried to *derive* gauge theory from first principles. The most important trait of [3] is that he asks the right questions from the outset, as to what happens when a Lagrangian invariant with respect to a global Lie group G is required to become invariant with respect to the local group $G(x)$. What kind of new (gauge) fields need be introduced to 'maintain' the symmetry? What is the form of the new Lagrangian, including the interaction? His answer is that the gauge field *must* be a spacetime vector field on which $G(x)$ acts by the adjoint representation, transforming in such a way that a covariant derivative exists. To our knowledge, the Utiyama argument is reproduced only in a couple of modern texts; such are [5] and [6]. I have profited from the excellent notes [7] as well.

One can speculate that, if the sequence of events had been slightly different, more attention would have been devoted to the theoretical underpinnings of the accepted dogma. It is revealing, and another pity, that Utiyama's later book in Japanese on the general gauge theory has never been translated.

1.2 The Utiyama analysis, first part

The starting point for Utiyama's analysis is a Lagrangian

$$\mathcal{L}(\varphi_k, \partial_\mu \varphi_k),$$

depending on a multiplet of fields φ_k and their first derivatives, *globally* invariant under a group G (of "gauge transformations of the first class") with n independent parametres θ^a. The group is supposed to be compact. We denote by f^{abc} the structure constants of its Lie algebra $\mathfrak{g}$; that is $\mathfrak{g}$ possesses generators T^a with commutation relations

$$[T^a, T^b] = f^{abc} T^c, \quad \text{with} \quad f^{abc} = -f^{bca},$$

and the Jacobi identity:

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \tag{1}$$

holds. We assume that the T^a can be chosen in such a way that f^{abc} is antisymmetric in all the three indices. This means that the adjoint representation of $\mathfrak{g}$ is semisimple, that is, $\mathfrak{g}$ is reductive [8, Chapter 15]. Close by the identity, an element $g \in G$ is of the form $\exp(T^a \theta^a)$.

The invariance is to be extended to a group $G(x)$ —of "gauge transformations of the second class"— depending on local parametres $\theta^a(x)$, in such

a way that a new Lagrangian $\mathcal{L}(\varphi_k, \partial_\mu \varphi_k, A)$ invariant under the wider class of transformations is uniquely determined. Utiyama's questions are:

- What new field $A(x)$ needs to be introduced?
- How does $A(x)$ transforms under $G(x)$?
- What are the form of the interaction and the new Lagrangian?
- What are the allowed field equations for $A(x)$?

The global invariance is given to us under the form:

$$\delta\varphi_k(x) = T^a_{kl}\varphi_l(x)\theta^a; \quad \text{now we want to consider} \quad \delta\varphi_k(x) = T^a_{kl}\varphi_l(x)\theta^a(x), \tag{2}$$

for $1 \leq a \leq n$. This last transformation in general does not leave $\mathcal{L}$ invariant. Let us first learn about the constraints imposed on the Lagrangian density by the assumed global invariance. One has

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_k}\,\delta\varphi_k + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_k)}\,\delta\,\partial_\mu\varphi_k, \tag{3}$$

where now

$$\delta\,\partial_\mu\varphi_k = \partial_\mu\,\delta\varphi_k = T^a_{kl}\partial_\mu\varphi_l(x)\theta^a(x) + T^a_{kl}\varphi_l(x)\partial_\mu\theta^a(x). \tag{4}$$

With a glance back to (3) and (4), we see that

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_k)}T^a_{kl}\varphi_l(x)\partial_\mu\theta^a(x) \neq 0. \tag{5}$$

Then it is necessary to add new fields $A'_p, p = 1, \dots, M$ in the Lagrangian, a process which we write as

$$\mathcal{L}(\varphi_k, \partial_\mu\varphi_k) \longrightarrow \mathcal{L}'(\varphi_k, \partial_\mu\varphi_k, A'_p).$$

The question is, how do the new fields transform? We assume not only a term of the form (4) but also a derivative term in $\theta^a(x)$ —indeed the latter will be needed to compensate the right hand side of (5):

$$\delta A'_p = U^a_{pq}A'_q\theta^a + C^{a\mu}_p\partial_\mu\theta^a. \tag{6}$$

Here $C^{a\mu}_p$ and the U^a_{pq} are constant matrices, for the moment unknown. The requirement is

$$0 = \delta\mathcal{L}' = \frac{\partial\mathcal{L}'}{\partial\varphi_k}\,\delta\varphi_k + \frac{\partial\mathcal{L}'}{\partial(\partial_\mu\varphi_k)}\,\partial_\mu\delta\varphi_k + \frac{\partial\mathcal{L}'}{\partial A'_p}\,\delta A'_p,$$

boiling down to

$$\delta\mathcal{L}' = \left[\frac{\partial\mathcal{L}'}{\partial\varphi_k}T^a_{kl}\varphi_l + \frac{\partial\mathcal{L}'}{\partial(\partial_\mu\varphi_k)}T^a_{kl}\partial_\mu\varphi_l + \frac{\partial\mathcal{L}'}{\partial A'_p}U^a_{pq}A'_q\right]\theta^a + \left[\frac{\partial\mathcal{L}'}{\partial(\partial_\mu\varphi_k)}T^a_{kl}\varphi_l + \frac{\partial\mathcal{L}'}{\partial A'_p}C^{a\mu}_p\right]\partial_\mu\theta^a = 0. \tag{7}$$

The coefficients must vanish separately, as the θ^a an their derivatives are arbitrary. The coefficient of $\partial_\mu\theta^a$ gives $4n$ equations involving A'_p, and hence to determine the A' dependence uniquely one needs $M = 4n$ components. Furthermore, the matrix $C^{a\mu}_p$ must be nonsingular. We have then an inverse:

$$C^{a\mu}_p {C^{-1}}^a_{\mu q} = \delta_{pq}; \qquad {C^{-1}}^a_{\mu p} C^{b\nu}_p = \delta^\nu_\mu\delta^{ab}.$$

Define the gauge (potential) field

$$A^a_\mu = \frac{1}{g}{C^{-1}}^a_{\mu p}A'_p, \quad \text{with inverse} \quad A'_p = gC^{a\mu}_p A^a_\mu. \tag{8}$$

Before proceeding, note that (6) and (8) together imply

$$\delta A^a_\mu = \left({C^{-1}}^a_{\mu p}U^c_{pq}C^{b\nu}_q\right)A^b_\nu\theta^c + \frac{\partial_\mu\theta^a}{g} =: (S^a_\mu)^{cb\nu}A^b_\nu\theta^c + \frac{\partial_\mu\theta^a}{g}.$$

Clearly from (7) we have

$$\frac{\partial\mathcal{L}'}{\partial(\partial_\mu\varphi_k)}T^a_{kl}\varphi_l + \frac{1}{g}\frac{\partial\mathcal{L}'}{\partial A^a_\mu} = 0.$$

Hence only the combination (called the covariant derivative)

$$D_\mu\varphi_k := \partial_\mu\varphi_k - gT^a_{kl}\varphi_l A^a_\mu$$

occurs in $\mathcal{L}'(\varphi_k, \partial_\mu\varphi_k, A'_p)$, and we rewrite:

$$\mathcal{L}'(\varphi_k, \partial_\mu\varphi_k, A'_p) \longrightarrow \mathcal{L}''(\varphi_k, D_\mu\varphi_k).$$

Moreover, it follows

$$\begin{aligned}
\frac{\partial\mathcal{L}'}{\partial\varphi_k} &= \frac{\partial\mathcal{L}''}{\partial\varphi_k} - g\frac{\partial\mathcal{L}''}{\partial(D_\mu\varphi_l)}T^a_{lk}A^a_\mu; \\
\frac{\partial\mathcal{L}'}{\partial(\partial_\mu\varphi_k)} &= \frac{\partial\mathcal{L}''}{\partial(D_\mu\varphi_k)}; \\
\frac{\partial\mathcal{L}'}{\partial A'_p} &= -\frac{\partial\mathcal{L}''}{\partial(D_\mu\varphi_k)}T^a_{kl}\varphi_l{C^{-1}}^a_{\mu p}.
\end{aligned}$$

Now we look at the vanishing coefficient of θ^a occurring in $\delta\mathcal{L}'$ in (7). By use of the last set of equations:

$$
\begin{aligned}
0 &= \frac{\partial\mathcal{L}''}{\partial\varphi_k}T^a_{kl}\varphi_l - g\frac{\partial\mathcal{L}''}{\partial(D_\mu\varphi_m)}T^b_{mk}T^a_{kl}A^b_\mu\varphi_l \\
&+ \frac{\partial\mathcal{L}''}{\partial D_\mu\varphi_k}T^a_{kl}\partial_\mu\varphi_l - g\frac{\partial\mathcal{L}''}{\partial(D_\mu\varphi_m)}T^c_{ml}\varphi_l {C^{-1}}^c{}_{\mu p}U^a_{pq}C^{b\nu}_q A^b_\nu \\
&= \frac{\partial\mathcal{L}''}{\partial\varphi_k}T^a_{kl}\varphi_l + \frac{\partial\mathcal{L}''}{\partial D_\mu\varphi_k}T^a_{kl}D_\mu\varphi_l \\
&- g\frac{\partial\mathcal{L}''}{\partial(D_\mu\varphi_m)}\big[T^b_{mk}T^a_{kl}A^b_\mu\varphi_l - T^a_{mk}T^b_{kl}A^b_\mu\varphi_l + T^c_{ml}(S^c_\mu)^{ab\nu}A^b_\nu\varphi_l\big].
\end{aligned} \tag{9}
$$

We are come thus to the crucial (and delicate) point. It seems that the two first terms in (9) cancel each other by global invariance (!) if we identify

$$\mathcal{L}''(\varphi_k, D_\mu\varphi_k) = \mathcal{L}(\varphi_k, D_\mu\varphi_k).$$

Utiyama [3] writes here: "This particular choice of $\mathcal{L}''$ is due to the requirement that when the field A is assumed to vanish, we must have the original Lagrangian". It seems to me, however, that covariance of $D_\mu\varphi_k$ is implicitly required. The whole procedure is at least consistent: the vanishing of the last term in (9) allows us to identify

$$(S^c_\mu)^{ab\nu} = f^{abc}\delta^\nu_\mu.$$

This implies in the end

$$\delta A^a_\mu = f^{cba}A^b_\mu\theta^c + \frac{\partial_\mu\theta^a}{g}. \tag{10}$$

As a consequence we obtain that $D_\mu\varphi_k$ indeed is a covariant quantity, in the sense of (4):

$$
\begin{aligned}
\delta(D_\mu\varphi_k) &= \delta(\partial_\mu\varphi_k - gT^a_{kl}A^a_\mu\varphi_l) = \partial_\mu(T^a_{kl}\theta^a\varphi_l) - gf^{cba}T^a_{km}A^b_\mu\theta^c\varphi_m \\
&- T^a_{kl}\partial_\mu\theta^a\varphi_l - gT^b_{kl}T^c_{lm}A^b_\mu\theta^c\varphi_m = T^a_{kl}\theta^a\partial_\mu\varphi_l - gT^c_{kl}T^b_{lm}A^b_\mu\theta^c\varphi_m \\
&= T^a_{kl}\theta^a(D_\mu\varphi_l).
\end{aligned}
$$

(In summary, Utiyama's argument here looks a bit circular to us; but all is well in the end.)

1.3 Final touches to the Lagrangian

The local Lagrangian of the matter fields contains in the bargain the interaction Lagrangian between matter and gauge fields. The missing piece is the Lagrangian for the "free" A-field. Next we investigate its possible type. Call the sought for Lagrangian $\mathcal{L}_0(A^a_\nu, \partial_\mu A^a_\nu)$. The invariance (under the local group of internal symmetry) postulate together with (10) in detail says:

$$\begin{aligned} 0 = &\left[\frac{\partial \mathcal{L}_0}{\partial A^a_\nu} f^{cba} A^b_\nu + \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^a_\nu)} f^{cba} \partial_\mu A^b_\nu\right] \theta^c \\ &+ \left[\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^a_\nu)} f^{cba} A^b_\nu + \frac{1}{g}\frac{\partial \mathcal{L}_0}{\partial A^c_\mu}\right] \partial_\mu \theta^c \\ &+ \frac{1}{g}\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^c_\nu)}\, \partial_{\mu\nu}\theta^c. \end{aligned}$$

As the θ^c are arbitrary again, one concludes that

$$\frac{\partial \mathcal{L}_0}{\partial A^a_\nu} f^{cba} A^b_\nu + \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^a_\nu)} f^{cba} \partial_\mu A^b_\nu = 0, \tag{11}$$

$$\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^a_\nu)} f^{cba} A^b_\nu + \frac{1}{g}\frac{\partial \mathcal{L}_0}{\partial A^c_\mu} = 0, \tag{12}$$

$$\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^a_\nu)} + \frac{\partial \mathcal{L}_0}{\partial(\partial_\nu A^a_\mu)} = 0. \tag{13}$$

Introduce provisionally:

$$\mathcal{A}^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu.$$

Then (12) is rewritten

$$\frac{\partial \mathcal{L}_0}{\partial A^c_\mu} + 2g\frac{\partial \mathcal{L}}{\partial(\mathcal{A}^a_{\mu\nu})} f^{cba} A^b_\nu = 0.$$

It ensues that the only combination occurring in the Lagrangian is

$$F^c_{\mu\nu} := \mathcal{A}^c_{\mu\nu} - \tfrac{1}{2} g f^{abc}(A^a_\mu A^b_\nu - A^a_\nu A^b_\mu). \tag{14}$$

One may write then

$$\mathcal{L}_0(A^a_\nu, \partial_\mu A^a_\nu) = \mathcal{L}'_0(F^a_{\mu\nu}).$$

Parenthetically we note

$$F^a_{\mu\nu} + F^a_{\nu\mu} = 0.$$

Now,
$$\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A^a_\nu)} = 2\frac{\partial \mathcal{L}'_0}{\partial F^a_{\mu\nu}}; \qquad \frac{\partial \mathcal{L}_0}{\partial A^b_\mu} = 2\frac{\partial \mathcal{L}'_0}{\partial F^c_{\mu\nu}} f^{abc} A^a_\nu.$$
Thus, by use of (1), formula (11) means
$$\frac{\partial \mathcal{L}'_0}{\partial F^c_{\mu\nu}} f^{abc} F^a_{\mu\nu} = 0, \tag{15}$$
for $1 \leq b \leq n$. This is left as an exercise. Also, by use of the identity of Jacobi again, one obtains
$$\delta F^c_{\mu\nu} = f^{abc} F^b_{\mu\nu} \theta^a. \tag{16}$$
This is a covariance equation similar to (4); its proof is an exercise as well.

Equation (15) is as far as we can go with the general argument. The simplest Lagrangian satisfying this condition is the quadratic in $F^a_{\mu\nu}$ one:
$$\mathcal{L}_{\mathrm{YM}} := -\tfrac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \quad \text{implying} \quad F^a_{\mu\nu} = -\frac{\partial \mathcal{L}_{\mathrm{YM}}}{\partial(\partial_\mu A^a_\nu)}. \tag{17}$$
The last equation is consistent with (13). Note that $\delta \mathcal{L}_{\mathrm{YM}} = 0$ from (16) is obvious.

If now we define
$$J^{c\mu} = g f^{abc} \frac{\partial \mathcal{L}_{\mathrm{YM}}}{\partial(\partial_\mu A^a_\nu)} A^b_\nu, \tag{18}$$
then from (11) again:
$$\partial_\mu J^{a\mu} = 0; \tag{19}$$
and from (12):
$$\partial^\nu F^a_{\mu\nu} = J^a_\mu, \tag{20}$$
by use of the equations of motion in both cases.

Let us take stock of what we obtained.

- Formula (18) tells us that (in this non-nabelian case) a self-interaction current J_μ exists, and gives us an explicit expression for it.
- Equation (19) furthermore shows that the current is conserved. Such a conservation equation, involving ordinary derivatives instead of covariant ones, does not look very natural perhaps, and is not so easy to prove directly —see the discussion in [9, Section 12-1-2]. This is the content of Noether's second theorem as applied in the present context.

- We observe that (20) is the field equation in the absence of matter fields.

The full Lagrangian is $\mathcal{L}(\varphi_k, D_\mu\varphi_k) + \mathcal{L}'_{\mathrm{YM}}$. One can proceed now to verify the invariance of it under the local transformation group and study the corresponding conserved currents. It should be clear that the conserved currents arising from local gauge invariance are exactly those following from global gauge invariance. Left as exercise.

1.4 The electromagnetic field

We illustrate only with the simplest example, as our main purpose is to produce a 'counterexample' pretty soon. Let a Dirac spinor field of mass M be given:

$$\mathcal{L} = \tfrac{i}{2}[\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\,\gamma^\mu\psi] - \overline{\psi}M\psi.$$

(Borrowing the frequent notation $A\overleftrightarrow{\partial^\alpha}B = A\partial^\alpha B - (\partial^\alpha A)B$, one can write this as well as

$$\tfrac{i}{2}\overline{\psi}\overleftrightarrow{\partial_\mu}\gamma^\mu\psi - \overline{\psi}M\psi.)$$

This is invariant under the global abelian group of phase transformations

$$\overline{\psi}(x) \mapsto e^{i\theta}\psi(x); \quad \psi(x) \mapsto e^{-i\theta}\psi(x);$$

or, infinitesimally,

$$\delta\overline{\psi} = i\overline{\psi}\theta; \quad \delta\psi = -i\psi\theta.$$

This leads to the covariant derivatives

$$D_\mu\overline{\psi} = \partial_\mu\overline{\psi} - igA_\mu\overline{\psi}; \quad D_\mu\psi = \partial_\mu\psi + igA_\mu\psi.$$

In conclusion, the original Lagrangian gets an interaction piece $-g\overline{\psi}\gamma^\mu A_\mu\psi$; with invariance of the new Lagrangian thanks to $\delta A_\mu = \partial_\mu\theta/g$. The full locally invariant Lagrangian is

$$\tfrac{i}{2}[\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\gamma^\mu\psi] - g\overline{\psi}\gamma^\mu A_\mu\psi - \overline{\psi}M\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

One can find now the associated electromagnetic current. This is the last exercise of this section.

1.5 The original Yang-Mills field

Consider an isospin doublet of spinor fields:

$$\psi = (\psi_k) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

with free Lagrangian

$$\tfrac{i}{2}[\overline{\psi}_k \gamma^\mu \partial_\mu \psi_k - \partial_\mu \overline{\psi}_k \gamma^\mu \psi_k] - \overline{\psi}_k M \psi_k.$$

This is invariant under the global $SU(2)$ group; with σ^a denoting as usual the Pauli matrices:

$$\psi_k \mapsto e^{-ig\theta^a\sigma^a/2}\big|_{kl}\psi_l; \qquad \overline{\psi}_k \mapsto \overline{\psi}_l\, e^{ig\theta^a\sigma^a/2}\big|_{lk}.$$

Infinitesimally,

$$\delta\psi_k = T^a_{kl}\psi_l\theta^a, \quad \text{with} \quad T^a_{kl} = -\frac{ig}{2}\sigma^a_{kl}.$$

We have $f^{abc} = g\epsilon^{abc}$ for this group. The Lagrangian becomes gauge invariant through the replacement

$$\partial_\mu\psi_k \mapsto D_\mu\psi_k = \partial_\mu\psi_k + \frac{ig}{2}\sigma^a_{kl}\psi_l A^a_\mu;$$

That is, the triplet of vector fields is the gauge (potential) field, the number of gauge field components being equal to the number of symmetry generators. Note the slight difference in the introduction of the coupling constant of the gauge field with the spinor field and itself.

The full locally invariant Lagrangian is

$$\tfrac{i}{2}[\overline{\psi}_k \gamma^\mu \partial_\mu \psi_k - \partial_\mu \overline{\psi}_k \gamma^\mu \psi_k] - \overline{\psi}_k M \psi_k - \tfrac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{g}{2}\overline{\psi}_k\gamma^\mu\sigma^a_{kl}\psi_l A^a_\mu,$$

with $F^a_{\mu\nu}$ given by (14). The current

$$\begin{aligned} J^a_\mu &= -\frac{g}{2}\overline{\psi}_k\gamma^\mu\sigma^a_{kl}\psi_l - g\epsilon^{abc}A^c_\nu\big[\partial_\mu A^b_\nu - \partial_\nu A^b_\mu - \frac{g}{2}\epsilon^{bde}(A^d_\mu A^e_\nu - A^d_\nu A^e_\mu)\big] \\ &= -\frac{g}{2}\overline{\psi}_k\gamma^\mu\sigma^a_{kl}\psi_l - g\epsilon^{abc}A^c_\nu\big(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu\big) + g^2(A^a_\mu(AA) + A^c_\mu A^c_\nu A^a_\nu), \end{aligned}$$

with $AA := A^c_\nu\, A^{c\nu}$, is conserved.

2 Massive vector fields

2.1 What is wrong with the Proca field?

The starting point in relativistic quantum physics is Wigner's theory of particles [10] as positive-energy irreps of the Poincaré group with finite spin/helicity. The transition to local free fields is made through intertwiners between the Wigner representation matrices and the matrices of covariant Lorentz group representations. Therefore, following standard notations [11], the general form of a quantum field is

$$\begin{aligned}
\varphi_l(x) &= \varphi_l^{(-)}(x) + \varphi_l^{(+)}(x) \quad \text{with} \\
\varphi_l^{(-)}(x) &= (2\pi)^{-3/2} \sum_{\sigma,n} \int d\mu_m(k)\, u_l(k,\sigma,n) e^{-ikx} a(k,\sigma,n); \\
\varphi_l^{(+)}(x) &= (2\pi)^{-3/2} \sum_{\sigma,n} \int d\mu_m(k)\, v_l(k,\sigma,n) e^{ikx} a^\dagger(k,\sigma,n);
\end{aligned}$$

with $d\mu_m(k)$ the usual Lorentz-invariant measure on the mass m hyperboloid in momentum space and n standing for particle species. Leaving the latter aside, the other labels are of representation-theoretic nature. Operator solutions to the wave equations carry the following labels, in all: the Poincaré representation (m,s), that gives the the mass shell condition and the spin s; the (k,σ), with the range of σ determined by s, label the momentum basis states; the (u,v) are Lorentz representation labels, usually appearing as a superscript indicating the tensorial or spinorial character of that solution. The c-number functions u_l, v_l in the plane-wave expansion formulae are the coefficient functions or intertwiners, connecting the set of creation or absorption operators $a^\#(k,\sigma)$, transforming as the irreducible representation (m,s) of the Poincaré group, to the set of field operators $\varphi_l(x)$, transforming as a certain finite-dimensional —thus nonunitary— irrep of the Lorentz group. We have thus in the vector field case

$$\begin{aligned}
\varphi^{(-)\mu}(x) &= (2\pi)^{-3/2} \sum_{\sigma} \int d\mu_m(k)\, u^\mu(k,\sigma) e^{-ikx} a(k,\sigma); \\
\varphi^{(+)\mu}(x) &= (2\pi)^{-3/2} \sum_{\sigma} \int d\mu_m(k)\, v^\mu(k,\sigma) e^{ikx} a^\dagger(k,\sigma).
\end{aligned}$$

We neglect to consider in the notation any colour quantum number for a while.

For the spin of the particle described by the vector field both the values $j=0$ and $j=1$ are possible. In the first case, at $\vec{k}=0$ only u^0, v^0 are

non-zero, and, dropping the label σ, we have by Lorentz invariance

$$u^\mu(k) \propto ik^\mu; \quad v^\mu(k) \propto -ik^\mu,$$

and therefore $\varphi^\mu(x) = \partial^\mu\varphi(x)$ for some scalar field φ. In the second case, only the space components u^j, v^j are not vanishing at $\vec{k} = 0$, and we are led to

$$\varphi^{(-)\mu}(x) = \varphi^{(+)\mu\dagger}(x) = (2\pi)^{-3/2} \sum_{\sigma=1}^{3} \int d\mu_m(k)\, \epsilon^\mu(k,\sigma) e^{-ikx} a(k,\sigma), \quad (21)$$

with ϵ^μ suitable (spacelike, normalized, orthogonal to k_μ, also real) polarization vectors, so that

$$\sum_{\sigma=1}^{3} \epsilon_\mu(k,\sigma)\epsilon_\nu(k,\sigma) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}. \quad (22)$$

On the right hand side we have the projection matrix on the space orthogonal to the four vector k^μ. This may be rewritten

$$\sum_{\sigma=0}^{\sigma=3} g_{\sigma\sigma}\epsilon_\mu(k,\sigma)\epsilon_\nu(k,\sigma) = g_{\mu\nu},$$

with the definition $\epsilon_\mu(k,0) = k_\mu/m$. With this treatment, we have the equations

$$(\Box + m^2)\varphi^\mu(x) = 0; \qquad \partial_\mu\varphi^\mu(x) = 0.$$

The last one ensures that one of the four degrees of freedom in φ^μ is elimimated. However, eventually (22) leads to the commutation relations for the Proca field of the form

$$[\varphi^\mu(x), \varphi^\nu(y)] = i\left(g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{m^2}\right)D(x-y).$$

In momentum space this is constant as $|k| \uparrow \infty$, which bodes badly for renormalizability. The Feynman propagator is proportional to

$$\frac{g_{\mu\nu} - k_\mu k_\nu/m^2}{k^2 - m^2};$$

there is moreover a troublesome extra term, that we leave aside.

The argument for non-renormalizability is as follows. Suppose that, as in the exampls of the previous section, the vector field is coupled with a conserved current made out of spinor fields. Consider an arbitrary Feynman

graph with E_F external fermion lines, I_F internal ones, and respectively E_B, I_B boson lines. The assumption says two fermion lines and one boson line meet at each vertex. The number of vertices is thus

$$V = 2I_B + E_B = \tfrac{1}{2}(2I_F + E_F).$$

Since there is a delta function for each vertex, one of them corresponding to overall momentum conservation, and each internal line has an integration over its moment, by eliminating I_F, I_B the superficial degree of divergence is

$$D = -4(V-1) + 3I_F + 4I_B = 4 + V - 3E_F/2 - 2E_B.$$

This shows that, no matter how many external lines are, the degree of divergence can be made arbitrarily large.

The difficulty is with the intertwiners, whose dimension does not allow to usual renormalizability condition. The idea is then to cure this by a cohomological extension of the Wigner representation space for massive spin 1 particles. This involves both the Stückelberg field and the ghost fields, already at the level of the description of free fields. The nilpotency condition $s^2 = 0$ for the BRS operator s will yield a cohomological representation for the physical Hilbert space $\ker s/\operatorname{ran} s$, which, as we shall see later, is the (closure of) the space of transversal vector wavefunctions. On that extended Hilbert space the renormalizability problem fades away. This goes in hand with a philosophy of primacy of a quantum character for the gauge principle, that should be read backwards into classical field theory; fibre bundle theory is no doubt elegant, but not intrinsic from this viewpoint. (For massless particles, the situtation is worse in that problematic aspects of the use of vector potentials in the local description of spin 1 particles show up already in the covariance properties of photons and gluons.)

2.2 What escaped through the net

Another unsung hero of quantum field theory is the Swiss physicist Ernst Carl Gerlach Stückelberg, barón von Breidenbach. He found himself among the pioneers of the 'new' Quantum Mechanics; at the end of the twenties, while working in Princeton with Morse, he was the one to explain the continuous spectrum of molecular hydrogen. At his return to Europe in 1933, he met Wentzel and Pauli for the first time. Stückelberg stayed in Zurich for two years before accepting a position at Genève. He turned to particle physics, where he will among other things contribute, according to his obituary [12], the meson hypothesis (unpublished at the time because of Pauli's criticism, and usually associated with Yukawa), the causal propagator (better known

as the Feynman propagator) and the renormalization group [13, 14]. Also by Stückelberg, not underlined in [12], are the first formulation of baryon number conservation; the first sketch of what is called nowadays Epstein–Glaser renormalization [15] —towards which, according to the account in [16], Pauli was better disposed— and the *Stückelberg field* [17], which concerns us here.

We have seen the extreme care that Utiyama put in deriving the precise form of gauge theory as a theorem. However, already at the moment that he published it, his result was false. That something that escapes through Utiyama's net is Stückelberg's gauge theory for massive spin 1 particles.

In the old paper [18] Pauli rather dismissively had given a short account of that before plunging into the Proca field; although anyone who has tried to work with the latter rapidly realizes it is good for nothing. There are several natural ways to discover the Stückelberg gauge field, even after one has been miseducated by textbooks —like [11]— into exclusively learning about the Proca field. A principled quantum approach is contained *in embrio* in the paper [19], where the starting point is Wigner's picture of the unitary irreps of the Poincaré group. In the book by Itzykson and Zuber, the Stückelberg method is used time and again [9, pp. 136, 172, 610] to smooth the $m \downarrow 0$ limit and exorcise infrared troubles. A very useful reference for the Stückelberg field is the review [20]. We have been inspired also by [21].

2.3 The Stückelberg field and Utiyama's test

Actually, there is no logical fault in the Lagrangian approach by Utiyama. Where he goes astray is only in the "initial condition" (2). We next try to find the Stückelberg field by the Utiyama path; that is, whether we actually could have derived the existence of the field B using the arguments of subsection 1.2. We do this for an abelian theory. Assume that a globally $G \equiv U(1)$-invariant model of a Dirac fermion of mass M and a real vector field of mass m are given:

$$\begin{aligned}\mathcal{L}_0 &= \frac{i}{2}(\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\,\gamma^\mu\psi) - \overline{\psi}M\psi + \tfrac{1}{2}m^2A_\mu A^\mu + \mathcal{L}_{\rm kin}(\partial_\nu A_\mu)\\ &=: \mathcal{L}_{0,\rm f} + \mathcal{L}_{0,\rm phmass} + \mathcal{L}_{\rm kin},\end{aligned}$$

with an obvious notation. This is obviously a model for (non-interacting) massive photon electrodynamics. Here $\mathcal{L}_{\rm kin}$ is the kinetic energy term for the photon, of the form (17). This Lagrangian is invariant under the global gauge transformations:

$$A_\mu(x) \mapsto A_\mu(x); \quad \overline{\psi}(x) \mapsto e^{i\theta}\psi(x); \quad \psi(x) \mapsto e^{-i\theta}\psi(x);$$

or, infinitesimally,

$$\delta A_\mu = 0; \quad \delta\overline{\psi} = i\overline{\psi}\theta; \quad \delta\psi = -i\psi\theta.$$

Now the Utiyama questions come in: what new (gauge) fields need be introduced? How do they transform under $G(x)$? What is the form of the interaction, and what is the new Lagrangian? To save spacetime, we restart from

$$\begin{aligned}&\tfrac{i}{2}[\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\gamma^\mu\psi] - \overline{\psi}\gamma^\mu A_\mu\psi - \overline{\psi}M\psi + \tfrac{1}{2}m^2 A_\mu A^\mu \\ &- \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) =: \mathcal{L}_{\mathrm{f}} + \mathcal{L}_{0,\mathrm{phmass}} + \mathcal{L}_{\mathrm{kin}}.\end{aligned}$$

The multiplet of fields includes now

$$\varphi = \begin{pmatrix} \overline{\psi} \\ \psi \\ A^\mu \end{pmatrix} \quad \text{transforming as} \quad \delta\varphi = \begin{pmatrix} i\overline{\psi}\theta(x) \\ -i\psi\theta(x) \\ \partial^\mu\theta(x) \end{pmatrix}; \tag{23}$$

where of course we required a variation of the QED type for the A_μ. For simplicity we have put $g = 1$. However, still

$$\delta\mathcal{L}_0 = \frac{\partial\mathcal{L}_{0,\mathrm{phmass}}}{\partial A_\mu}\,\delta A_\mu = m\partial_\mu\theta \neq 0.$$

It seems that, when vector fields are conjured *ab initio*, further infinitesimal gauge transformations of the form

$$\delta\varphi_k = \mathcal{A}_{kc}\theta_c + \mathcal{B}^\nu_{kc}\partial_\nu\theta_c, \tag{24}$$

need to be considered. Here we have a particular case, with a trivial colour index c; with $\varphi_k \to A_\mu$; $\mathcal{A}_\mu$ vanishing; and $\mathcal{B}^\nu_\mu = \delta^\nu_\mu$.

There is no need to involve other parts of the Lagrangian than $\mathcal{L}_{0,\mathrm{phmass}}$ in the remaining calculation. We need an extra vector field. It is natural to think that it be fabricated from the derivatives of a scalar B, and we write:

$$\mathcal{L}_{0,\mathrm{phmass}}(A_\mu) \longrightarrow \mathcal{L}'(A_\mu, \partial_\mu B).$$

It is immediate to note that if we assume the new field transforms like $\delta B = m\theta$, then the requirement of local gauge invariance is

$$\delta\mathcal{L}' = \left[\frac{\partial\mathcal{L}'}{\partial A_\mu} + m\frac{\partial\mathcal{L}'}{\partial(\partial_\mu B)}\right]\partial_\mu\theta = 0.$$

It follows

$$m\frac{\partial\mathcal{L}'}{\partial(\partial_\mu B)} = -\frac{\partial\mathcal{L}'}{\partial A_\mu}.$$

Consequently only the combination

$$A_\mu - \partial_\mu B/m$$

occurs in $\mathcal{L}'(A_\mu, \partial_\mu B)$. Thus we rewrite:

$$\mathcal{L}'(A_\mu, \partial_\mu B) \longrightarrow \mathcal{L}_{0,\mathrm{phmass}}(A_\mu - \partial_\mu B/m).$$

The bosonic part of the Lagrangian is *in fine*

$$\mathcal{L}_\mathrm{b} = \mathcal{L}_\mathrm{kin} + \frac{m^2}{2}\left(A_\mu - \frac{\partial_\mu B}{m}\right)^2;$$

note that, with $V_\mu = (A_\mu - \partial_\mu B/m)$, one has $\mathcal{L}_\mathrm{kin}(A_\mu) = \mathcal{L}_\mathrm{kin}(V_\mu)$. The total Lagrangian $\mathcal{L} = \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{b}$ has what we want. With the multiplet of fields

$$\varphi = \begin{pmatrix} \overline{\psi} \\ \psi \\ A^\mu \\ B \end{pmatrix} \quad \text{transforming as} \quad \delta\varphi = \begin{pmatrix} i\overline{\psi}\theta(x) \\ -i\psi\theta(x) \\ \partial^\mu\theta(x) \\ m\theta(x) \end{pmatrix},$$

we plainly obtain local gauge invariance of $\mathcal{L}_\mathrm{f}, \mathcal{L}_\mathrm{b}$ and $\mathcal{L}$. Note the Euler–Lagrange equation

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial_\mu B} = \frac{\partial\mathcal{L}}{\partial B} \quad \text{yielding} \quad \Box B = m\,\partial A.$$

Note as well that one can fix the gauge so B vanishes; this does not mean the gauge symmetry is trivial.

Maybe Utiyama missed this because [22] he only takes into account, for the original variables, infinitesimal gauge transformations typical of 'matter' fields, of the form (4); he did not consider the possibility (23), that is (10), for the vector fields acting as sources of gauge fields.

We finish this subsection by noting that $\mathcal{L}_\mathrm{b}$ may be written as well

$$\mathcal{L}_\mathrm{b} = (\partial_\mu - igA_\mu)\Phi\,(\partial_\mu + igA_\mu)\Phi^*, \quad \text{with} \quad \Phi = \frac{m}{\sqrt{2}g}\exp(igB/m);$$

that is an abelian Higgs model without self-interaction. The verification is straightforward.

2.4 The Stückelberg formalism for non-abelian Yang–Mills fields

The sophisticated method for this was established by Kunimasa and Goto [23]; we follow in the main [24]. For apparent simplicity, consider an isovector field A_μ^a interacting with an isospinor spinor field ψ, like in subsection 1.5. Let us choose the notation

$$\mathbb{A}_\mu = \tfrac{1}{2}\sigma^a A_\mu^a; \qquad \mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig(\mathbb{A}_\mu \mathbb{A}_\nu - \mathbb{A}_\nu \mathbb{A}_\mu).$$

Indeed $\frac{i}{4}\sigma^a\sigma^b = -\frac{1}{2}\epsilon^{abc}\sigma^c$, in consonance with (14). The Lagrangian density is written

$$-\tfrac{1}{2}\operatorname{tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}) + \tfrac{i}{2}\overline{\psi}\overleftrightarrow{\partial_\mu}\gamma^\mu\psi - \overline{\psi}M\psi - g\overline{\psi}\gamma^\mu A_\mu\psi.$$

This is invariant under

$$\psi \to \mathbb{W}^{-1}\psi; \qquad \mathbb{A}_\mu \to \mathbb{W}^{-1}\mathbb{A}_\mu\mathbb{W} - \frac{i}{g}\mathbb{W}^{-1}\partial\mu\mathbb{W},$$

for $\mathbb{W} \in SU(2)$; which is nothing but (10), with

$$\mathbb{W} = \exp(T^a\theta^a(x)).$$

To make the mass term

$$m^2\operatorname{tr}(\mathbb{A}_\mu\mathbb{A}^\mu) = \tfrac{1}{2}m^2 A_\mu^a A^{a\mu}$$

gauge invariant, it is enough to introduce a 2×2 matrix ω_μ of auxiliary vector fields, so that

$$m^2\operatorname{tr}(\mathbb{A}_\mu - \omega_\mu/g)$$

is invariant under gauge transformations, if

$$\omega_\mu \to \mathbb{W}^{-1}\omega_\mu\mathbb{W} - i\mathbb{W}^{-1}\partial_\mu\mathbb{W}. \tag{25}$$

Indeed, let $C \in SU(2)$ transform as $C \to C\mathbb{W}$. Then

$$\omega_\mu := -iC^{-1}\partial_\mu C$$

satisfies (25):

$$-i\mathbb{W}^{-1}C^{-1}\partial_\mu C\mathbb{W} = \mathbb{W}^{-1}\omega_\mu\mathbb{W} - i\mathbb{W}^{-1}\partial_\mu\mathbb{W}.$$

With $C = \exp(B^aT^a/m)$, we can think of the B^a as the auxiliary fields.

We may add, however, that the introduction of scalar Stückelberg partners for the A_μ^a by the substitution $\mathbb{A}_\mu \to \mathbb{A}_\mu - \partial_\mu B$, with $B = B^aT^a$, seems to work as well. In gauge theory, the "elegant" non-infinitesimal notation is a bit dangerous, in that it tends to obscure the fact that the transformation of the gauge fields (10) is *independent* of the considered representation of the gauge group.

2.5 Gauge-fixing and the Stückelberg Lagrangian

We begin to face quantization now. For that, we need to fix a gauge. Otherwise, we cannot even derive a propagator from the Lagrangian. Let us briefly recall the standard argument:

$$\mathcal{L}_{\mathrm{QED}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}A^\mu \mathcal{D}_{\mu\nu}A^\nu,$$

with

$$\mathcal{D}_{\mu\nu}(x) = -g_{\mu\nu}\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma} + \overleftarrow{\partial_\nu}\overrightarrow{\partial_\mu} \quad \text{or} \quad \mathcal{D}_{\mu\nu}(k) = -g_{\mu\nu}k^2 + k_\mu k_\nu,$$

in momentum space. The matrix $\mathcal{D}_{\mu\nu}$ has null determinant and thus is not invertible; so one cannot define a Feynman propagator. This is precisely due to gauge invariance. The same problem for QED was cured by Fermi long ago [25] by introduction of the piece $\frac{-1}{2\alpha}(\partial^\nu A_\nu)^2$. Here we proceed similarly, and the gauge-fixing term we take is of the 't Hooft type:

$$\mathcal{L}_{\mathrm{gf}} = \frac{-1}{2\alpha}(\partial^\nu A_\nu + \alpha m B)^2. \tag{26}$$

We denote

$$\mathcal{L}_S = \mathcal{L} + \mathcal{L}_{\mathrm{gf}},$$

the Stückelberg Lagrangian. The gauge-fixing amounts to that now the gauge variation θ must satisfy the Klein–Gordon equation with mass $m\sqrt{\alpha}$:

$$(\Box + \alpha m^2)\theta = 0;$$

just like in old trick by Fermi in electrodynamics, where the new Lagrangian is still gauge-invariant provided we assume $\Box\theta = 0$ for the gauge variations. Now instead the Euler–Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}_S}{\partial(\partial_\mu B)} = \frac{\partial \mathcal{L}_S}{\partial B} \quad \text{yields} \quad (\Box + \alpha m^2)B = 0.$$

Hence the gauge-fixing implies B itself now is a free field with mass $m\sqrt{\alpha}$. Another good reason for the gauge-fixing is to keep A_ν as an honest-to-God spin 1 field in the interaction. Recall that in a quantum vector field spin 0 and 1 are possible. The scalar B 'extracts' the spin 0 part, so the remaining part is transverse. In fact $\partial_\mu(A^\mu - \partial^\mu B/m) = \partial A + \alpha m B$ if the equation of motion is taken into account; and this gauge-fixing term is destined to vanish in an appropriate sense on the physical state space.

A word is needed on the Noether theorem now. There is now an extra term in $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$, of the form $-\frac{g^{\mu\nu}}{\alpha}(\partial A + \alpha m B)$. This gives rise to the Euler–Lagrange equation:

$$\Box A_\mu + \Big(\frac{1}{\alpha} - 1\Big)\partial_\mu(\partial A) + m^2 A_\mu = g\overline{\psi}\gamma_\mu\psi, \tag{27}$$

where we have restablished temporarily the coupling constant. As a consequence of (27) we have

$$\Box\, \partial A + \alpha m^2 \partial A = 0.$$

The simplest option now is to take $\alpha = 1$ (so the masses of A_ν and B coincide), as then the A_ν obey the Klein–Gordon equation at zeroth order in g. This could be termed the 'Feynman gauge'. But in some contexts it is important to keep the freedom of different mass values for the vector and the scalar bosons. (We have for the fermion the Dirac equation

$$i\gamma^\mu \partial_\mu \psi = (g\gamma^\mu A_\mu + M)\psi,$$

and its conjugate. Nothing new here.)

A comment on renormalizability is in order at this point. The choice $\alpha \downarrow 0$ is the Landau gauge, in which renormalizability is almost explicit. On the other hand, it is clear that $B = 0$ (the original Proca model), where the theory is non-renormalizable by power counting, can be recovered as a sort of 'unitary gauge'. If we can prove gauge covariance of the theory, all these versions will be physically equivalent. An extra advantage of the Stückelberg field in renormalization is that, because it cures the limit $m \downarrow 0$, it allows the use of masses as infrared regulators.

To finish, we call the attention again upon the similitudes of the model with the abelian Higgs model. Upon renormalization, a "Higgs potential-like" term pops up in the Lagrangian. However, the vacuum expected value of the Stückelberg field is still zero. For non-abelian theories, the situation remains murky even now.

2.6 The ghosts we called over

For completeness, we insert next a conventional discussion of BRS invariance for the Lagrangian obtained in the previous subsection. (This is not intended to be discussed during the lessons, and both the cognoscenti and the non-cognoscenti may skip it in first reading.)

Nowadays BRS invariance of the (final) Lagrangian is an integral part of the quantization process. Among other things, it helps to establish gauge *covariance*, that is, independence of the chosen gauge for physical quantities;

in turn this helps with renormalizability proofs. We approach the quantum context by introducing two fermionic ghosts $\omega, \tilde{\omega}$ plus an auxiliar (Nakanishi–Lautrup) field h that we add to the collection φ. From the infinitesimal gauge transformations we read off the BRS transformation:

$$s\varphi = s \begin{pmatrix} \overline{\psi} \\ \psi \\ A^\mu \\ B \\ \omega \\ \tilde{\omega} \\ h \end{pmatrix} = \begin{pmatrix} i\omega\overline{\psi} \\ -i\omega\psi \\ \partial^\mu\omega \\ m\omega \\ 0 \\ h \\ 0 \end{pmatrix}.$$

It is clear that s increases the ghost number by one. Extend s as an antiderivation; from the fact that $\omega, \tilde{\omega}$ are anticommuting we obtain (even off-shell) nilpotency of order two for the BRS transformation: $s^2 = 0$ (we will always understand 'nilpotent of order two' for 'nilpotent' in this work). Now, in the BRS approach, one takes the action to be a local action functional of matter, gauge, ghost and h-fields with ghost number zero and invariant under s. This is provided by the new form

$$\mathcal{L}_{\rm gf} = s[\mathcal{F}(\overline{\psi}, \psi, A_\mu, B)\tilde{\omega} + \tfrac{1}{2}\alpha h\tilde{\omega}],$$

for the gauge-fixing term of the Lagrangian. Here $\mathcal{F}$ is the gauge-fixing functional, like $(\partial^\mu A_\mu + \alpha m B)$ above. Invariance comes from $s\mathcal{L}_{\rm gf} = 0$ on account of nilpotency, of course. We can rewrite

$$\mathcal{L}_{\rm gf} = -\tilde{\omega} s\mathcal{F} + h\mathcal{F} + \tfrac{1}{2}\alpha h^2 = -\tilde{\omega} s\mathcal{F} + \tfrac{1}{2}\Big(\frac{\mathcal{F}}{\sqrt{\alpha}} + h\sqrt{\alpha}\Big)^2 - \frac{\mathcal{F}^2}{2\alpha}.$$

One can eliminate h using its equation of motion

$$0 = \frac{\partial \mathcal{L}_{\rm gf}}{\partial h} = \mathcal{F} + h\alpha, \quad \text{so that} \quad \mathcal{L}_{\rm gf} = -\tilde{\omega} s\mathcal{F} - \frac{\mathcal{F}^2}{2\alpha}.$$

and also $s\tilde{\omega} = -\mathcal{F}/\alpha$: the BRS transformation maps then the anti-ghosts or dual ghosts into the gauge-fixing terms (the price to pay is that s would be nilpotent off-shell only when acting on functionals independent of $\tilde{\omega}$). In our case (26):

$$s\mathcal{F} = s(\partial^\mu A_\mu + \alpha m B) = (\Box + \alpha m^2)\omega.$$

Thus the contribution of the fermionic ghosts in this abelian model to $\mathcal{L}_{\rm gf}$ is

$$-\tilde{\omega} s\mathcal{F} = -\tilde{\omega}(\Box + \alpha m^2)\omega;$$

also $\partial_\mu\tilde\omega\partial^\mu\omega - \tilde\omega\alpha m^2\omega$ would do; the ghosts turn out to be free fields with the same mass as Stückelberg's B-field. Notice that the ghost term decouples in the final effective Lagrangian. (According to [26], adding to the action a term invariant under the BRS transformation amounts to a redefinition of the fields coupled to the source in the generating functional; this has no influence on the $\mathbb{S}$-matrix.)

We have followed [8] and mainly [27] in this subsection.

At the end of the day, the Lagrangian for massive electrodynamics is of the form

$$\begin{aligned}
&\mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{kin} + \mathcal{L}_\mathrm{b} + \mathcal{L}_\mathrm{gf} = \tfrac{i}{2}[\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\gamma^\mu\psi] - \overline{\psi}\gamma^\mu A_\mu\psi - \overline{\psi}M\psi \\
&- \tfrac14(FF) + \frac{m^2}{2}(A - \partial B/m)^2 - \frac{1}{2\alpha}(\partial A + \alpha m B)^2 - \tilde\omega(\Box + \alpha m^2)\omega \\
&= \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{kin} + \frac{m^2A^2}{2} - \frac{1}{2\alpha}(\partial A)^2 + \tfrac12(\partial B)^2 - \frac{\alpha m^2}{2}B^2 - m\partial_\mu(BA^\mu) \\
&= -\tilde\omega(\Box + \alpha m^2)\omega.
\end{aligned}$$

Highlights:

- The gauge-fixing has been chosen independently of the matter field.
- The gauge sector contains first a massive vector field, with three physical components of mass m (one longitudinal and two transverse) and an unphysical spin-zero piece of mass $\sqrt{\alpha}m$.
- The cross term between A_μ and B disappeared.
- The gauge sector also contains a (commuting) Stückelberg B-field with mass $\sqrt{\alpha}m$ and a pair of (anticommuting) ghost-antighost scalars, with mass $\sqrt{\alpha}m$ as well.
- For computing $\mathbb{S}$-matrix elements, the ghosts can be integrated out, since they are decoupled and do not appear in asymptotic states. But we cannot integrate out the B-field, because, as discussed in Section 3, it plays a role in the definition of the physical states —and moreover it undergoes a non-trivial renormalization.
- The only interacting piece is the $\overline{\psi}A\psi$ term in the fermionic part of the Lagrangian.
- The model is *renormalizable.*

3 Quantization of massive spin-1 fields

3.1 On the need for BRS invariance

It is impossible for us, within the narrow limits of this short course, to follow in any meaningful detail the tortuous chronological path to the discovery of BRS invariance in relation with gauge invariance. The story in outline is well-known. By fixing the gauge, Feynman was able to generate Feynman diagrams [28] for non-abelian gauge theories; but unitarity of the $\mathbb{S}$-matrix was lost unless additional "probability-eating" quantum fields were introduced. The auxiliary ghost fields appeared clearly in the work by Fadeev and Popov, that uses the functional integral. In the seventies it was discovered that the resulting effective Lagrangian still supports a global invariance of a new kind, the nilpotent BRS transformation, that allows to recover unitarity, ensures gauge independence of the quantum observables and powerfully contributes to the proofs of renormalizability.

We attacked quantization in subsection 2.1 through the canonical method. So we motivate the introduction of the ghosts and BRS symmetry/operator in our previous considerations. Now that hopefully we have broken the mental association between "gauge principle" and "masslessness", one can proceed to a simple and general version of gauge theory with BRS invariance. The quantization of massive vector fields is interesting in that it is conceptually simpler, although analytically more complicated, than that of massless ones. (It is true that in theories with massive gauge bosons, the masses are generated by the 'Higgs mechanism'; but this is just a poetic description that cannot be verified or fasified at present.) In the context, concretely we need the ghosts as "renormalization catalysers". In fact, it has been shown in [19] that for interacting massive vector field models the renormalizability condition fixes the theory completely, including the cohomological extension of the Wigner representation theory by the ghosts, and the Stückelberg field in the abelian case —even if you had never heard of it in a semi-classical study of Lagrangians, like the one performed in Section 2. As well as a Higgs-like field for flavourdynamics; we shall touch upon this in the last section.

The crucial problem, illustrated by our discussion in subsection 2.1, is to eliminate the unphysical degrees of freedom in the quantization of free vector fields in a subtler way than Proca's, particularly without giving up commutators of the form

$$[A_\mu(x), A_\nu(y)] = ig_{\mu\nu}D(x-y), \quad A_\mu^+ = A_\mu. \tag{28}$$

Also we ask for the KG equations $(\Box + m^2)A^\mu = 0$ to hold (in the Feynman gauge). It is impossible to realize (28) on Hilbert space. Let us sketch

the solution in this subsection. It goes through the introduction of a distinguished symmetry η (that is, an operator both selfadjoint and unitary), called the Krein operator, on the Hilbert–Fock space H. Whenever such a Krein operator is considered, the η-adjoint O^+ of an operator O is defined:

$$O^+ = \eta O^\dagger \eta.$$

Let $(.,.)$ denote the positive definite scalar product in H. Then

$$\langle .,. \rangle := (.,\eta .)$$

gives an 'indefinite scalar product', and the definition of O^+ is just that of the adjoint with respect to $\langle .,. \rangle$. The algebraic properties are like in usual adjugation $\dagger$, but O^+O is not positive in general.

The pair (H, η), where H is the original Hilbert–Fock space, including ghosts, is called a Krein space. The undesired contributions from the A-space will be cancelled by the 'unphysical' statistics of the ghosts. The BRS operator is an (unbounded) nilpotent η-selfadjoint operator Q on H. That is, $Q^2 = 0, Q = Q^+$. By means of Q one shows that H (or a suitable dense domain of it) splits in the direct sum of three pairwise orthogonal subspaces (quite analogous to the Hodge–de Rham decomposition in differential geometry of manifolds):

$$H = \operatorname{ran} Q \oplus \operatorname{ran} Q^\dagger \oplus (\ker Q \cap \ker Q^\dagger).$$

In addition we assume

$$\eta\Big|_{\ker Q \cap \ker Q^\dagger} = 1.$$

That is, $\langle .,. \rangle$ is positive definite on

$$H_{\text{phys}} := \ker Q \cap \ker Q^\dagger,$$

which is called the physical subspace. An alternative definition for H_{phys} is the cohomological one:

$$H_{\text{phys}} = \ker Q / \operatorname{ran} Q.$$

Nilpotency of Q is the reason to introduce the anticommuting pair of ghost fields. In interaction, the $\mathbb{S}$-matrix must be physically consistent:

$$[Q, \mathbb{S}]_+ = 0, \quad \text{or at least} \quad [Q, \mathbb{S}]_+\Big|_{\ker Q} = 0.$$

In the following subsections we flesh out the details of all this.

3.2 Ghosts as free quantum fields

A first step in a rigorous construction of ghosts is their understanding as quantum fields, together with the issue of the 'failure' of the spin-statistics theorem for them. We look for two operator-valued distributions $u, \tilde{u}$, acting on a Hilbert–Fock space H_{gh} and satisfying Klein–Gordon (KG) equations:

$$(\Box + m^2)u = (\Box + m^2)\tilde{u} = 0, \tag{29}$$

and the following commutation relations, in the sense of tempered distributions

$$[u_a(x), \tilde{u}_b(y)]_+ = -i\delta_{ab}D(x-y); \qquad [u_a(x), u_b(y)]_+ = [\tilde{u}_a(x), \tilde{u}_b(y)]_+ = 0.$$

Here $D = D^+ + D^-$ is the Jordan–Pauli function; we refer to the supplement at the end of these notes for notation regarding the propagators. The fields 'live' in the adjoint representation of a gauge group G (as the gauge fields themselves); the colour indices a, b most often can be omitted. The components of H_{gh} of degree n are *skewsymmetric* square-summable functions (with the Lorentz-invariant measure $d\mu_m(p)$) of n momenta on the mass hyperboloid $\mathcal{H}_m$, with their colour indices and ghost indices, where the first, say a, can run from 1 to $\dim G$, and we let the second, say i, take the values ± 1. (The reader is warned of that the notation for the ghost fields in this section, and a few other notational conventions, are different from we found convenient in the sections dealing with the semi-classical aspects.)

We proceed to the construction. Consider the dense domain $\mathcal{D} \subset H_{\mathrm{gh}}$ of vectors with finitely many nonvanishing components which are Schwartz functions of their arguments. Then there exist the annihilation (unbounded) operator functions $c_{a,i}(p)$ of $\mathcal{D}$ into itself, given by

$$[c_{a,i}(p)\Phi]^{(n)}_{a_1,\dots,a_n;i_1,\dots,i_n}(p_1,\dots,p_n) = \sqrt{n+1}\,\Phi^{(n+1)}_{a,a_1,\dots,a_n;i,i_1,\dots,i_n}(p,p_1,\dots,p_n).$$

Integrating this with a Schwartz function on the mass hyperboloid gives a bounded operator. The adjoint of $c_{a,i}(p)$ is defined as a sesquilinear form on $\mathcal{D} \otimes \mathcal{D}$, and we have the usual "commutation relations" among them:

$$[c_{a,i}(p), c^\dagger_{b,j}(p')]_+ = \delta_{ab}\delta_{ij}\delta(p-p');$$

otherwise zero. Notice that $\delta(p-p')$ is shorthand for the Lorentz invariant Dirac distribution $2E\delta(\vec{p}-\vec{p}\,')$ corresponding to $d\mu_m(p)$.

We are set now to define the distributional ghost field operators in coordinate space out of the $c_{a,i}, c^\dagger_{b,j}$. The construction is diagonal in the G-index, so it will be omitted. The general Ansatz is

$$u_i(x) = \int d\mu_m(p)\,\bigl[A_{ij}c_j(p)e^{-ipx} + B_{ij}c^\dagger_j(p)e^{+ipx}\bigr].$$

Here

$$A = \begin{pmatrix} A_{11} & A_{1-1} \\ A_{-11} & A_{-1-1} \end{pmatrix}; \qquad B = \begin{pmatrix} B_{11} & B_{1-1} \\ B_{-11} & B_{-1-1} \end{pmatrix}.$$

Since p is on the mass hyperboloid the KG equations (29) hold. The anticommutators are:

$$[u_i(x), u_j(y)]_+ = -i\big[A_{ik}B_{jk}D^+(x-y) - B_{ik}A_{jk}D^-(x-y)\big].$$

The only combinations with causal support are multiples of $D^+ + D^-$. As we want to keep causality, it must be $AB^t + BA^t = 0$, so we obtain

$$[u_i(x), u_j(y)]_+ = -iC_{ij}D(x-y),$$

with $C := AB^t$ skewsymmetrical. There are of course many possible choices of A, B with this constraint. We pick:

$$C = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

This finally gives:

$$u(x) = u_1(x) = \int d\mu_m(p)\, \big(c_1(p)e^{-ipx} + c^\dagger_{-1}(p)e^{ipx}\big);$$
$$\tilde{u}(x) = u_{-1}(x) = \int d\mu_m(p)\, \big(c_{-1}(p)e^{-ipx} - c^\dagger_1(p)e^{ipx}\big).$$

We remark $[\tilde{u}(x), u(y)]_+ = iD(x-y) = -iD(y-x) = [u(y), \tilde{u}(x)]_+$.

The representation of the Poincaré group is the same as for $2\dim G$ independent scalar fields; we do not bother to write it. As we have chosen A, B invertible, the creation and annihilation operators can be expressed in terms of the ghost fields and their adjoints. Then the vacuum is cyclic with respect to these.

Defining the adjoint fields, one sees that the anticommutators of the ghost fields with their adjoints are not causal. This, according to [29, 30] allows to escape the spin-statistics theorem. Indeed, a version of the last says that no nonvanishing scalar fields can exist satisfying

$$[u_a(x), u_b(y)]_+ = 0, \qquad [u_a(x), u^\dagger_b(y)]_+ = 0,$$

for spacelike separations. Because the second anticommutator is not causal, the last condition is not violated. (There are other explanations in the literature for the same conumdrum, though.)

3.3 Mathematical structure of BRS theories

There are several questions relative at the scheme proposed in 3.1, that we address systematically now.

1. What is the algebraic framework?
2. In which mathematical sense BRS invariance is a symmetry?
3. When is there a BRS charge associated to a BRS symmetry?
4. What are the continuity properties of the generator Q?
5. How the 'Hodge–de Rham' decomposition of the Hilbert space takes place?
6. How are the physical states characterized?

The first paper to tackle these questions was the famous on the quark confinement problem by Kugo and Ojima [31], although their answers were not quite correct. A very good treatment, that we follow for the most part, was given by Horuzhy and Voronin [32].

1. Consider a 'general BRS theory' on a Krein space (H, η). On a suitable common invariant dense domain $\mathcal{D} \subset H$ there is defined a system of physical quantum fields and ghost fields (the physical fields could be matter fields, Yang–Mills fields or, say, the coordinates of a first-quantized string), forming a polynomial algebra $\mathcal{A}$; the operator id $\in \mathcal{A}$ on H we denote by 1. A Krein operator has the eigenvalues ± 1, so $\eta = P^\eta_+ - P^\eta_-$ with an obvious notation. We assume moreover $\dim P^\eta_\pm H = \infty$. By O° we shall mean the restriction of O^+ to $\mathcal{D}$. We say O is η-selfadjoint when $O = O^\circ$; η-unitary when $O^{-1} = O^\circ$. The field algebra has a cyclic vector or 'vacuum' $|0\rangle$, that is, $\mathcal{A}|0\rangle$ is dense in $\mathcal{D}$.

2. Mathematically speaking, a BRS (infinitesimal) transformation is a skew-adjoint, nilpotent superderivation s acting on the field algebra of H. Let $\epsilon_O := (-)^{\mathbb{N}_{\rm gh}(O)}$, whith $\mathbb{N}_{\rm gh}(O)$ the number of ghost fields in the monomial O. Typically s changes the ghost number by one. Then s is a linear map of $\mathcal{A}$ into $\mathcal{A}$ such that
$$s(OB) = s(O)B + \epsilon_O O s(B),\ s^2 = 0,\ \epsilon_{s(O)} = -\epsilon_O$$
$$\text{and} \quad s(O)^\circ = -\epsilon_O s(O^\circ).$$
The key point for BRS invariance is obviously the nilpotency equation $s^2 = 0$.

3. An important question is whether the BRS transformation s possesses a generator or BRS charge Q, that is, takes the form

$$s(O) = [Q, O]_\pm \quad \text{where} \quad [Q, O]_\pm := QO - \epsilon_O OQ. \tag{30}$$

Indeed, we may try to equivalently write (30) as

$$QO|0\rangle = s(O)|0\rangle.$$

This equation will serve as definition of Q, at least on a dense subset of $\mathcal{D}$, provided

$$O|0\rangle = 0 \quad \text{implies} \quad s(O)|0\rangle = 0.$$

Note that $Q|0\rangle = 0$ because $s(1) = 0$. Thus (30) is consistent. Nilpotency of Q follows:

$$Q^2 O|0\rangle = Qs(Q)|0\rangle = 0.$$

One expects Q as defined above to be η-selfadjoint. But this is not completely automatic. We have

$$\begin{aligned}\langle QO|0\rangle, B|0\rangle\rangle &= \langle s(O)|0\rangle, B|0\rangle\rangle = \langle B^\circ s(O)|0\rangle, |0\rangle\rangle \\ &= \epsilon_{B^\circ}\langle\big(s(B^\circ O) - s(B^\circ)O\big)|0\rangle, |0\rangle\rangle \\ &= \epsilon_O\langle 0 \mid s(O^\circ B) \mid 0\rangle \\ &+ \langle O|0\rangle, s(B)|0\rangle\rangle.\end{aligned} \tag{31}$$

This will be equal to $\langle O|0\rangle, QB|0\rangle\rangle$ if in general we have

$$\langle 0 \mid s(O) \mid 0\rangle = 0 \quad \text{for all} \quad O \in \mathcal{A}.$$

In this case, we have η-symmetry. For passing to η-selfadjointness, consult [33].

Reciprocally, if Q is η-selfadjoint with $Q|0\rangle = 0$, nilpotent, and generates s by (30), then, rather trivially:

$$\langle 0 \mid s(O) \mid 0\rangle = \langle 0 \mid [Q, O]_\pm) \mid 0\rangle = \langle Q|0\rangle, O|0\rangle\rangle = 0.$$

Moreover, for s so defined

$$\begin{aligned} s(O)^\circ &= (QO - \epsilon_O OQ)^\circ = O^\circ Q - \epsilon_O QO^\circ \\ &= -\epsilon_O(QO^\circ - O^\circ Q) = -\epsilon_O s(O^\circ).\end{aligned}$$

We finally verify nilpotency of s:

$$s^2(O) := [Q, [Q, O]_\pm]_\pm = Q(QO - \epsilon_O OQ) + \epsilon_O(QO - \epsilon_O OQ)Q = 0.$$

4. In physics Q is often treated as a bounded operator. But there are large classes of nilpotent, η-selfadjoint unbounded operators. Let for instance $H = H_1 \oplus H_2$ and

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

with A unbounded and skewadjoint. Then Q is nilpotent, η-selfadjoint and unbounded. For another example, take $H = H_1 \otimes H_2$, where H_1 is an infinite-dimensional Hilbert space, H_2 is a Krein space, $Q = O \otimes B$, with $O = O^\dagger$ unbounded and B nilpotent and η-selfadjoint. Typically BRS operators are sums of such operators.

Given an arbitrary nilpotent operator Q, such that $\operatorname{dom} Q^2$ is dense, the following holds: either Q is bounded, with 0 as unique point in its spectrum, or Q is unbounded and its spectrum is all of the complex plane.

Proof. Assume $\operatorname{spec} Q \neq \mathbb{C}$. Let λ belong to the resolvent of Q. Then Q is closed, as $Q - \lambda$ is. (We recall that a Hilbert space operator is by definition closed when its graph is closed. Also by definition, $Q - \lambda$ is a one-to-one map from $\operatorname{dom} Q$ onto H with bounded inverse, so it is closed.) Now $(Q - \lambda)^{-1} H \subset \operatorname{dom} Q$. Therefore

$$(Q - \lambda)\big(Q + \lambda Q (Q - \lambda)^{-1}\big)$$

makes sense and is equal to Q^2. Now Q is closed and $\lambda Q (Q - \lambda)^{-1}$ is bounded, therefore $Q + \lambda Q (Q - \lambda)^{-1}$ is closed; then Q^2 is closed. Therefore its domain is all of H, so Q is bounded (by the closed graph theorem). Then it is well known that $\operatorname{spec} Q = \{ 0 \}$. $\square$

5. Consider the subspaces $\ker Q, \eta \ker Q, \operatorname{ran} Q, \eta \operatorname{ran} Q$. Due to $Q^2 = 0$, we can assume $\operatorname{ran} Q \subset \operatorname{dom} Q$; otherwise we extend Q to the whole $\operatorname{ran} Q$ by zero. Because of η-selfadjointness, $\ker Q$ is closed; also, $\eta \operatorname{ran} Q = \eta \operatorname{ran} \eta Q^\dagger \eta = \operatorname{ran} Q^\dagger$ and $\eta \ker Q = \ker \eta Q \eta = \ker Q^\dagger$. In view of nilpotency, it is immediate that

$$\operatorname{ran} Q \perp \operatorname{ran} Q^\dagger,$$

where $\perp$ indicates perpendicularity in the Hilbert space sense. We have

$$(\operatorname{ran} Q \oplus \operatorname{ran} Q^\dagger)^\perp = \ker Q^\dagger \cap \ker Q.$$

Indeed, the domain of $Q^\dagger$ is dense in H and thus $(x, Q^\dagger y) = 0$ for all $y \in \operatorname{dom} Q^\dagger$ implies $Qx = 0$. Similarly for $(\operatorname{ran} Q)^\perp = \ker Q^\dagger$. Denoting by $[\perp]$ perpendicularity in the Krein space sense, it also clear that

$$(\ker Q^\dagger \cap \ker Q)^\perp = (\ker Q^\dagger \cap \ker Q)^{[\perp]}$$

In summary

$$\begin{aligned} H &= \overline{\operatorname{ran} Q^\dagger} \oplus \ker Q = \overline{\operatorname{ran} Q} \oplus \ker Q^\dagger = \overline{\operatorname{ran} Q} \oplus \overline{\operatorname{ran} Q^\dagger} \oplus (\ker Q^\dagger \cap \ker Q) \\ &= \overline{\operatorname{ran} Q} \oplus \overline{\operatorname{ran} Q^\dagger} [+] (\ker Q^\dagger \cap \ker Q); \end{aligned} \tag{32}$$

where the last symbol means the η-orthogonal sum. This is the 'Hodge–de Rham' decomposition of H.

6. Assume moreover

$$\eta\Big|_{\ker Q \cap \ker Q^\dagger} = 1.$$

Then we baptize

$$H_{\text{phys}} := \ker Q \cap \ker Q^\dagger,$$

the physical subspace, on which $\langle ., . \rangle$ is positive. Alternative characterizations are

$$H_{\text{phys}} = \ker Q / \overline{\operatorname{ran} Q},$$

in view of (32), and

$$H_{\text{phys}} = \ker [Q, Q^\dagger]_+.$$

Indeed $[Q, Q^\dagger]_+ \, x = 0$ iff $Qx = Q^\dagger x = 0$.

3.4 BRS theory for massive spin one fields

We finally turn to our physical case. When dealing with the massive vector field, instead of eliminating ab initio the longitudinal component as in (21), we keep the $a(k, 0)$ and their adjoints, and proceed as follows. We recognize Krein spaces as appropriate tools to study (quantum) gauge theories. In our present case $\eta := (-)^{\mathbb{N}_l}$, where $\mathbb{N}_l$ is the particle number operator for the longitudinal modes. Now

$$A^\mu(x) = (2\pi)^{-3/2} \sum_{\sigma=0}^{3} \int d\mu_m(k) \, \big(\epsilon^\mu(k, \sigma) e^{-ikx} a(k, \sigma) + \epsilon^\mu(k, \sigma) e^{ikx} a^+(k, \sigma) \big).$$

Clearly

$$a^+(k, 0) = -\eta^2 a^\dagger(k, 0) = -a^\dagger(k, 0);$$

however, by definition $A^\mu(x)$ is η-selfconjugate.

We hasten to indicate the main difference with the massless case. Note that a unitary representation of the Poincaré group on the original space is given by

$$U(a,\Lambda)A^\mu(x)U^{-1}(a,\Lambda) = \Lambda^\mu_\nu A^\nu(\Lambda x + a) = U^{-1+}(a,\Lambda)A^\mu(x)U^+(a,\Lambda).$$

This implies

$$[U^+(a,\Lambda)U(a,\Lambda), A^\mu(x)] = 0;$$

therefore U is η-unitary. As $\mathbb{N}_l$, thus η, commutes with U —basically because the longitudinal polarization transforms into itself under a Lorentz transformation,

$$\Lambda^\nu_\mu \epsilon^\mu(k,0) = \frac{(\Lambda k)^\nu}{m} = \epsilon^\nu(\Lambda k, 0),$$

the representation U is also unitary. This cannot be obtained in the massless case.

The commutation relations for A-field are of the form

$$[A^\mu(x), A^\nu(y)] = ig^{\mu\nu}D(x-y),$$

as we wished for. We now employ a nilpotent gauge charge Q to characterize the physical state subspace and eliminate the unphysical longitudinal mode. For photons, the definition of Q is known to be

$$Q = \int_{x^0=\mathrm{const}} d^3x\,(\partial\cdot A)\overleftrightarrow{\partial_0}u. \tag{33}$$

Let us accept this is a conserved quantity, associated to the current

$$j_\mu = (\partial\cdot A)\overleftrightarrow{\partial_\mu}u.$$

Obviously $[Q,u]=0$. By use of the algebraic identity

$$[AB,C]_+ = A[B,C]_+ - [A,C]B,$$

nilpotency then is checked as follows:

$$2Q^2 = [Q,Q]_+ = -\int_{x^0=\mathrm{const}} d^3x\,[(\partial\cdot A),Q]\overleftrightarrow{\partial_0}u = i\int_{x^0=\mathrm{const}} d^3x\,\Box u\overleftrightarrow{\partial_0}u = 0,$$

because the ghost is a free massless quantum field, ie, satifies the wave equation. The form (33) will not do for the massive case, as now, with ghost

fields of the same mass as A^μ, after a relatively long calculation involving the solution of the Cauchy problem for u, we would obtain

$$2Q^2 = i\int_{x^0=\mathrm{const}} d^3x\,\Box u\,\overleftrightarrow{\partial_0}\,u = -im^2\int_{x^0=\mathrm{const}} d^3x\,u\,\overleftrightarrow{\partial_0}\,u \neq 0,$$

A suitable form of Q is reached by introducing a (Bose) scalar field with the same mass, satisfying

$$(\Box + m^2)B = 0, \quad [B(x), B(y)] = -iD(x-y),$$

and then

$$Q = \int_{x^0=\mathrm{const}} d^3x\,(\partial\cdot A + mB)\,\overleftrightarrow{\partial_0}\,u. \tag{34}$$

We leave to the care of the reader to check this is a conserved quantity. Now we obtain

$$2Q^2 = i\int_{x^0=\mathrm{const}} d^3x\,\Box u\,\overleftrightarrow{\partial_0}\,u + im^2\int_{x^0=\mathrm{const}} d^3x\,u\,\overleftrightarrow{\partial_0}\,u = 0.$$

In this way we have recovered the Stückelberg field!

In summary, the gauge variations are:

$$\begin{aligned} sA^\mu(x) &= [Q, A^\mu(x)]_\pm = i\partial^\mu u(x);\\ sB(x) &= [Q, B(x)]_\pm = imu(x);\\ su(x) &= [Q, u(x)]_\pm = 0;\\ s\tilde{u}(x) &= [Q, \tilde{u}(x)]_\pm = -i\big(\partial^\mu A_\mu(x) + mB(x)\big); \end{aligned} \tag{35}$$

with respect to the semi-classical analysis in Section 2 there is a slight change of notations; the present ones are more advantageous when dealing with quantum fields. As expected, the BRS variation of the gauge field corresponds to susbtituting the ghost field for the infinitesimal parameter of the gauge transformation.

We finish by a little collection of remarks.

- The ghost number of Q is precisely 1.
- In view of nilpotency of Q, finite gauge variations are easily computed. We have

  $$A'_\mu(x) = e^{-i\lambda Q}A_\mu(x)e^{i\lambda Q} = A_\mu(x) - i\lambda[Q, A_\mu(x)] - \tfrac{1}{2}\lambda^2[Q,[Q,A_\mu]].$$

  Note that the last term is *not* zero. But certainly there are no higher-order terms.

- Only unphysical fields appear in the formula (34) for Q.

- A stronger BRS theory includes the anti-BRS symmetry $\bar{s}$, with the 'complete nilpotency' conditions $s^2 = \bar{s}^2 = s\bar{s} + \bar{s}s = 0$ [34]. The main role of $\bar{s}$ is to ensure the closure of the classical algebra, at the level of Lagrangians. This is more or less unnecessary in Yang–Mils theories, but useful for instance in supersymmetric theories.

- It would seem that the foregoing analysis applies only to abelian fields. The cognoscenti would in general expect in formula (35) extra terms in the first equality (covariant derivative rather than ordinary one) and in the the third one (a ghost term involving the structure constants). That is:

$$\begin{aligned} sA_\mu^a(x) &= [Q, A_\mu^a(x)] = iD_\mu u^a(x); \\ su^a(x) &= [Q, u^a(x)]_+ = -\tfrac{i}{2} g f^{abc} u^b(x) u^c(x); \end{aligned} \tag{36}$$

  However, it ain't necessarily so. By just adding the colour index, one can think of (35) as a first step, one in which self-interaction is neglected, for a non-abelian theory. In the causal approach to QFT [30], one approaches interacting fields by means of free fields, and then both methods differ.

3.5 The ghostly Krein operator

For completeness, we include here a discussion on the "charge algebra" for ghosts. Let f_r denote an orthonormal basis of $L^2(\mathcal{H}_m, d\mu_m(p))$. Consider the charge operators

$$Q(A) := \sum_{r,b,i} c^\dagger_{b,i}(f_r) a_{ij} c_{b,j}(f_r) = \sum_{b,i} \int d\mu_m(p)\, c^\dagger_{b,i}(p) a_{ij} c_{b,j}(p),$$

for $A = (a_{ij})$ a $2{\times}2$ matrix. This is defined on a common dense domain of $H_{\rm gh}$, bigger than $\mathcal{D}$, which is mapped by the charge operators into itself. This map represents $\mathfrak{gl}(2, \mathbb{C})$, as

$$Q(AB - BA) = Q(A)Q(B) - Q(B)Q(A); \quad \text{also} \quad Q(A^\dagger) = Q^\dagger(A).$$

By the way, by $Q^\dagger(A)$ we mean its restriction to $\mathcal{D}$. Taking for A the unit matrix and the Pauli matrix σ_3, we respectively obtain the ghost number $\mathbb{N}_{\rm gh}$

and ghost charge Q_{gh} operators. The other two Pauli matrices yield ghost-antighost exchanging operators, respectively called here Γ, Ω. Their commutators with the local fields $u, \tilde{u}$ are:

$$\begin{aligned}
[\mathbb{N}_{\mathrm{gh}}, u] &= -\tilde{u}^\dagger, \qquad [\mathbb{N}_{\mathrm{gh}}, \tilde{u}] = u^\dagger; \\
[Q_{\mathrm{gh}}, u] &= -u, \qquad [Q_{\mathrm{gh}}, \tilde{u}] = \tilde{u}; \\
[\Gamma, u] &= \tilde{u}, \qquad [\Gamma, \tilde{u}] = u; \\
[\Omega, u] &= -i\tilde{u}, \qquad [\Omega, \tilde{u}] = iu.
\end{aligned}$$

The verification of this is an exercise. The generator of $\mathbb{N}_{\mathrm{gh}}$, that constitutes the centre of the charge algebra, gives by commutation with $u, \tilde{u}$ not relatively local fields. We write down the following currents:

$$\begin{aligned}
j_{\mathbb{N}_{\mathrm{gh}}}(x) &:= i{:}u^\dagger(x)\overleftrightarrow{\partial^\mu}u(x){:}; \quad j_{\mathrm{gh}}(x) := i{:}\tilde{u}(x)\overleftrightarrow{\partial^\mu}u(x){:}; \\
j_u(x) &:= i{:}u(x)\overleftrightarrow{\partial^\mu}u(x){:}; \quad j_{\tilde{u}}(x) := i{:}\tilde{u}(x)\overleftrightarrow{\partial^\mu}\tilde{u}(x){:}.
\end{aligned}$$

Again, $j_{\mathbb{N}_{\mathrm{gh}}}$ is not a relatively local quantum field. They are related to the corresponding charges in the usual way; one has, moreover

$$\Gamma = \tfrac{1}{2}(Q_u - Q_{\tilde{u}}), \quad \Omega = \tfrac{i}{2}(Q_u + Q_{\tilde{u}}).$$

We can consider as well operators $T\big(e^{iA}\big) := \exp(iQ(A))$. They give a representation of the general linear group. It is $T(B^\dagger) = T(B)^\dagger$. Also $T(B)Q(A)T^{-1}(B) = Q(BAB^{-1})$.

The theory with ghosts has to be constructed by using only the fields $u, \tilde{u}$, while their adjoints will not appear at all; in this way the troubles with locality are avoided. In massless Yang–Mills theories, say, one considers the interaction

$$T_1(x) = \tfrac{i}{2}f^{abc}\big({:}A^a_\mu A^b_\nu F^{c\mu\nu}{:}(x) + {:}A^a_\mu u^b \partial^\mu \tilde{u}^c{:}\big)(x). \tag{37}$$

This is invariant under gauge transformations generated by the differential operator (33). The $u^\dagger, \tilde{u}^\dagger$ do not appear here. But then it is right to worry about unitarity. The solution in gauge theories is as follows: η-unitarity of $\mathbb{S}$ together with gauge invariance will imply unitarity of the $\mathbb{S}$-matrix on the 'physical subspace'.

For the theory defined by (37), we have

$$\eta = \eta_A \otimes \eta_{\mathrm{gh}} \quad \text{on} \quad H = H_A \otimes H_{\mathrm{gh}}.$$

We recall η_A is given by

$$\eta_A = \prod_{a=1}^{\dim G} (-)^{\mathbb{N}_{0a}},$$

where $\mathbb{N}_{0a}$ is the number operator for gauge particles of G-colour a. The gauge potentials A^a_μ are η-hermitian. Grosso modo: we expect the η-adjoint fields $u^+, \tilde{u}^+$ to enter T_1, in order to have $\eta_{\rm gh}$-hermitian quantities. The key is causality: the latter Krein operator must be defined in a way that $u^+, \tilde{u}^+$ are relatively local to $u, \tilde{u}$; we know $u^\dagger, \tilde{u}^\dagger$ do not have this property. With all this in mind, we search for the 'good' $\eta_{\rm gh}$. Clearly, it cannot be relatively local itself, which is tantamount to involve $\mathbb{N}_{\rm gh}$. A natural guess would be to take the (already much used) operator:

$$E := \exp(i\pi\mathbb{N}_{\rm gh}).$$

However, consider the ghost and antighost number operators:

$$N_j := \tfrac{1}{2}(\mathbb{N}_{\rm gh} + jQ_{\rm gh}),$$

for $j = 1, -1$. They also have integer spectrum. Moreover:

$$E = (-)^{N_1+N_{-1}} = (-)^{N_1-N_{-1}} = (-)^{Q_{\rm gh}},$$

so E cannot be the right choice. We consider instead

$$I := (-)^{N_{-1}} = e^{\frac{i}{2}\pi(N-Q_{\rm gh})} = T(\sigma_3).$$

This is indeed a symmetry. We do have $Ic_j(p)I = jc_j(p)$, and it is then quickly seen that

$$Iu^\dagger I = \tilde{u}; \quad I\tilde{u}^\dagger I = u;$$

so we have locality. While this is a perfectly sensible solution to the problem, T_1 and Q are not I-hermitian. One could write different, equivalent expressions for the terms involving ghosts in the Lagrangian (see the discussion in the next paragraph); but first we submit to convention. Consider then

$$\begin{aligned} S = T(U) &:= T\big(i(\sigma_1+\sigma_3)/\sqrt{2}\big) = T\Big(e^{i\pi(\sigma_1+\sigma_3)/2\sqrt{2}}\Big) \quad \text{and} \\ \eta_{\rm gh} &:= SIS^{-1} = T(\sigma_1) = i^{N-\Gamma}. \end{aligned}$$

Now we get:

$$\eta_{\rm gh}c_j(p)\eta_{\rm gh} = c_{-j}(p),$$

and

$$u^+ := \eta_{\rm gh}u^\dagger\eta_{\rm gh} = u; \quad \tilde{u}^+ := \eta_{\rm gh}\tilde{u}^\dagger\eta_{\rm gh} = -\tilde{u};$$

together with

$$T_1^+ = T_1; \qquad Q^+ = Q.$$

An alternative definition for the ghost contribution in T_1 would be given by

$$\tfrac{1}{2} f^{abc} {:} A^a_\mu u^b \overleftrightarrow{\partial^\mu} \tilde{u}^c {:}(x) \quad \text{instead of} \quad f^{abc} {:} A^a_\mu u^b \partial^\mu \tilde{u}^c {:}(x).$$

Both forms differ by a pure divergence term plus a Q_{gh}-coboundary, that is, a term of the form $[Q_{\mathrm{gh}}, K]_+$. Therefore the first one remains gauge invariant. The choice of it would allow the use of I as Krein operator, preserving all the good properties. The second one is employed partly for historical reasons.

To conclude, let us comment again on the different behaviour of the Poincaré group representation in the massive and the massless case. For the former, the representation is always unitary, and commutes with all charges $Q(A)$ and transformations $T(B)$. Therefore it is η-unitary as well. However, for the gauge potentials in the massless case the representation is not unitary, and η_A is introduced for reasons of covariance.

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